

Exercise 31

Solve the boundary-value problem, if possible.

$$y'' + 4y' + 20y = 0, \quad y(0) = 1, \quad y(\pi) = 2$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad y' = re^{rx} \quad \rightarrow \quad y'' = r^2e^{rx}$$

Plug these formulas into the ODE.

$$r^2e^{rx} + 4(re^{rx}) + 20(e^{rx}) = 0$$

Divide both sides by e^{rx} .

$$r^2 + 4r + 20 = 0$$

Solve for r .

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(20)}}{2} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i$$

$$r = \{-2 - 4i, -2 + 4i\}$$

Two solutions to the ODE are $e^{(-2-4i)x}$ and $e^{(-2+4i)x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1e^{(-2-4i)x} + C_2e^{(-2+4i)x} \\ &= C_1e^{-2x}e^{-4ix} + C_2e^{-2x}e^{4ix} \\ &= e^{-2x}(C_1e^{-4ix} + C_2e^{4ix}) \\ &= e^{-2x}[C_1(\cos 4x - i \sin 4x) + C_2(\cos 4x + i \sin 4x)] \\ &= e^{-2x}[(C_1 + C_2) \cos 4x + (-iC_1 + iC_2) \sin 4x] \\ &= e^{-2x}(C_3 \cos 4x + C_4 \sin 4x). \end{aligned}$$

Apply the boundary conditions to determine C_3 and C_4 .

$$y(0) = C_3 = 1$$

$$y(\pi) = e^{-2\pi}C_3 = 2$$

Solving this system of equations yields $C_3 = 1$ and $C_3 = 2e^{2\pi}$, which is a contradiction. Therefore, there's no solution to the boundary value problem.